Imaging Atoms in a Lattice v.2

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Introduction

Purpose of this Report

The question to try in answer is "what will we see when we look at atoms in an optical lattice?". This will depend on a number of different factors - the number of adjacent 2D planes occupied, the number of photons we are able to collect, etc. However, the primary information we are trying to obtain is the fidelity of identifying an atom's presence on a lattice site (or not) as a function of our imaging system's numerical aperture (NA) and aberrations.

Theory

We start from the Debye approximation (Born and Wolf, p. 487) for the 3D Electric field near the focus of an imaging lens:

$$E\left(x,y,z\right) = -iA\frac{2\pi}{\lambda}\left(\frac{a}{f}\right)^2 e^{i\left(\frac{f}{a}\right)^2 u}\int_0^1 J_0\left(v\rho\right)e^{-iu\rho^2}\rho d\rho \tag{1}$$

where u and v are the scaled axial and radial coordinates, respectively:

$$\begin{aligned} u &= \frac{2\pi}{\lambda}\left(\frac{a}{f}\right)^2 z \\ v &= \frac{2\pi}{\lambda}\left(\frac{a}{f}\right)\sqrt{x^2+y^2} \end{aligned} \tag{2}$$

f is the effective focal length,
a is the radius of the aperture,
λ is the wavelength of light being collected,
A is an amplitude,
$J_0\left(x\right)$ is the zeroth order Bessel function, and
ρ is a scaled aperture coordinate that we integrate over.
The intensity can then be written:

$$I(x,y,z) = \frac{c\epsilon_0}{2} \| E(x,y,z) |^2 \tag{3}$$

See Figure 1 for an image.

Atomic system

We have atoms in a cubic lattice with a lattice constant $d = 532nm$ that fluoresce at $\lambda = 405nm$. Furthermore, we assume that the atoms lie in the ground state of each lattice site, with the lattice oscillation period during imaging $1/\omega_{latt} = V_{latt}\frac{2k^2}{m} \approx 50kHz$ much greater than the imaging time $t_{image} \approx 100ms$, such that we can represent the each atom's position as the convolution of a point source with a gaussian probability distribution:

$$I_\sigma(x,y,z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} \frac{1}{\sqrt{2\pi\sigma_y^2}} \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-\frac{x_0^2}{2\sigma^2}} e^{-\frac{y_0^2}{2\sigma^2}} e^{-\frac{z_0^2}{2\sigma^2}} \times I(x-x_0, y-y_0, z-z_0)\, dx_0 dy_0 dz_0 \tag{4}$$

where $\sigma = \sqrt{\frac{\hbar}{2m\omega_{latt}}}$ is the ground state harmonic oscillator length.

A caveat: I have done all calculations so far in Mathematica. At least under my handling, Mathematica is too slow to be able to calculate $I_\sigma(x,y,z)$ for more than a couple of atoms in any reasonable amount of time. Therefore, I have obtained all results by using the simplification $I_\sigma(x,y,z) = I(x,y,z)$. Yes, this increases the quality of the image artificially, but we can add in some position uncertainty later on (in the Visibility Variance section) to compensate. This, at least, seemed reasonable to me...

If atoms are illuminated incoherently, then we can write the total intensity (as seen through the objective) as:

$$I_{image}(x,y,z) = \sum_i \sum_j \sum_k I_\sigma(x-id, y-jd, z-kd) \tag{5}$$

where i, j and k label the atom positions in the x, y and z directions. Note that we are *not* restricting ourselves to the situation of a single occupied 2D plane (in which case $k = 0$ always).

Imaging system

For the purposes of this report, we assume that we will be able to observe atoms with a numerical aperture between $NA = 0.4 - 0.75$. We will also assume that the effective focal length of our imaging system can be approximated to be $f = 4mm$, as estimated from the Zeiss Plan NeoFluor 0.75 objective. We can then change the NA by changing the size of the aperture a according to:

$$NA = nsin\theta = nsin\left(arctan\left(\frac{a}{f}\right)\right) \tag{6}$$

To write $I(x,y,z)$ we need to know what a is in terms of NA:

$$a = \frac{f}{\sqrt{(n/NA)^2 - 1}}$$

Note that we are assuming a perfectly shaped point spread function (PSF) always, i.e. one that is an Airy disk. We will treat aberrations later on.

Pixellization

We capture light on a CCD camera with 6.45umx6.45um pixels. With 63x magnification, this corresponds to a pixel size of roughly 100nmx100nm at the object. Therefore, we capture a power on each pixel equal to:

$$P_{image}(x,y,z) = I_{image}(x,y,z) \cdot (100nm)^2$$

1 Visibility

To first (zeroth?) order, we will assume that the ability to distinguish an atom's presence on a site is related to the difference in intensities at a lattice site and midway to another - in the same focal plane, of course. Let's define this quantity the visibility V:

$$\begin{aligned} V &= \frac{P_{max} - P_{min}}{P_{max} + P_{min}} \\ &= \frac{P_{image}(0,0,0) - P_{image}(d/2,0,0)}{P_{image}(0,0,0) + P_{image}(d/2,0,0)} \end{aligned} \tag{7}$$

See Figure 3 for results.

2 Threshold Identification

While visibility might be a good measure for quantifying the quality of an image, it is not the most direct measure of what we actually want to know - whether there are zero or one atoms on a given site. The idea behind the "Threshold Identification" measure is as follows: If we measure a power P_{id} within an area centered upon a site of interest, then the presence of zero or one atoms can be determined by the criteria:

- If $P_{id} < P_0$, then there are no atoms on site.
- If $P_{1a} < P_{id} < P_{1b}$, then there is one atom on site.

where P_{id} is given by:

$$P_{id} = \sum_{i=-3}^{3} \sum_{j=-3}^{3} I_\sigma \left(x - i \cdot 100nm, y - j \cdot 100nm, z\right) \cdot \left(100nm\right)^2 \tag{8}$$

and the identification threshold powers P_0, P_{1a}, P_{1b}, and P_2 can be calculated from:

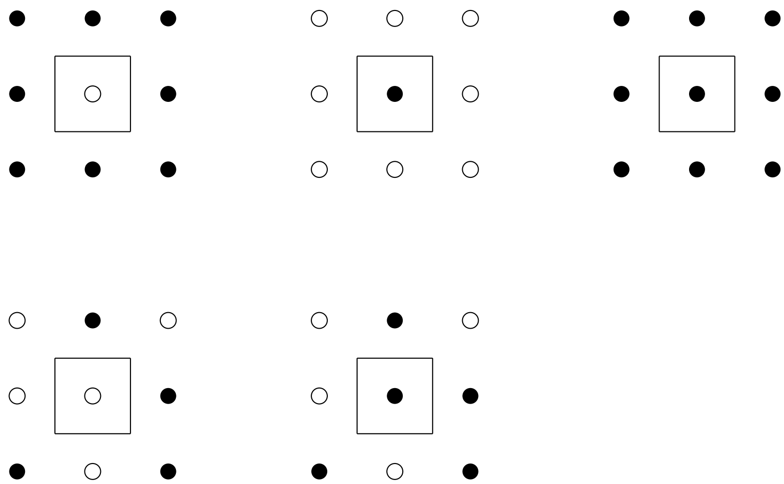
Worst case scenario

$$\begin{aligned}
P_0 &= \sum_{i=-3}^{3} \sum_{j=-3}^{3} I_\sigma \left(x - i \cdot 100nm, y - j \cdot 100nm, z; 0_{site}; 1_{adjacent}\right) \cdot \left(100nm\right)^2 \\
P_{1a} &= \sum_{i=-3}^{3} \sum_{j=-3}^{3} I_\sigma \left(x - i \cdot 100nm, y - j \cdot 100nm, z; 1_{site}; 0_{adjacent}\right) \cdot \left(100nm\right)^2 \\
P_{1b} &= \sum_{i=-3}^{3} \sum_{j=-3}^{3} I_\sigma \left(x - i \cdot 100nm, y - j \cdot 100nm, z; 1_{site}; 1_{adjacent}\right) \cdot \left(100nm\right)^2
\end{aligned} \tag{9}$$

Average scenario

$$\begin{aligned}
P_0 &= \sum_{i=-3}^{3} \sum_{j=-3}^{3} I_\sigma \left(x - i \cdot 100nm, y - j \cdot 100nm, z; 0_{site}; rand_{adjacent}\right) \cdot \left(100nm\right)^2 \\
P_1 &= \sum_{i=-3}^{3} \sum_{j=-3}^{3} I_\sigma \left(x - i \cdot 100nm, y - j \cdot 100nm, z; 1_{site}; rand_{adjacent}\right) \cdot \left(100nm\right)^2
\end{aligned} \tag{10}$$

where the subscript *site* identifies the occupancy on the site of interest, the subscript *adjacent* identifies the occupancy of the some number of adjacent sites, and *rand* is a random choice between 1 and 0. The worst case scenario imagines the hardest to distinguish occupation situations, whereas the average case scenario describes the half-filled situation. Results can be seen in Figure 4. Because the worst case scenario has a multiplicity of 1 and the average case has a multiplicity of 70 and 2E12 in the single and multi-plane case, respectively, the average case is more representative. Both are plotted.



Picture 1: Single-plane representation of worst-case occupation situations (*top*) and average case (*bottom*). On top, from left to right, we have P_0, P_{1a}, and P_{1b}. On bottom, from left to right, we have P_0 and P_1. The box surrounding the middle site is the measurement area.

Shot Noise

As mentioned before, there will be shot noise associated with each measurement on each pixel (which, as mentioned before, we will take to be approximately 100nmx100nm at the object) that will hinder our ability to distinguish 0 atoms from 1 atom on a site. If the most likely photon number is N, then the probability of capturing k photons is given by a Poissonian distribtuion:

$$Prob\left(N_i;k\right) = \frac{\left(N_i\right)^k e^{-N_i}}{k!}$$

where

$$N_0 = P_0 \cdot t_{imaging}$$

$$N_1 = P_1 \cdot t_{imaging}$$

Fidelity

We define the Fidelity as the certainty with which we are able to identify the presence, or not, of an atom on a lattice site:

$$F = 1 - \sum_{k=0}^{\infty} \sqrt{Prob\left(N_0;k\right) \cdot Prob\left(N_1;k\right)}$$

where $F = F\left(N_{scatter}, NA\right)$ is a function of the number of scattered photons $N_{scatter}$ and the numerical aperture NA. The results can be seen in Figure 6.

3 Matching and Minimizing

The "Matching and Minimizing" method also seeks to directly answer the question of a site's occupancy. It seems like a needless extension of the "Identification Power" method - requiring more, and more difficult, analysis while only reducing the uncertainty of the identification (which may or may not be necessary) - but I state the idea here anyways. Keep in mind I have no results. Therefore, this method does not provide yet a criteria to decide upon an objective.

The idea is as follows:

- As stated before, we have 3 possible occupancies for each site: zero, one, or two atoms
- We also know the position of site relative to all the rest: seperated by the lattice constant d
- Say we have n adjacent sites that contribute some non-negligible intensity to the site region of interest.
- We can then construct 3^n possible intensity distributions. Let's call the i^{th} distribution possibility $I_i(x, y, z)$.
- We then compare, or match, the measured intensity $I_{measured}(x, y, z)$ with each of the possible intensity distributions $I_i(x, y, z)$, identifying i from:

  $$|\alpha \cdot I_{measured}(x, y, z) - I_i(x, y, z)| = 0$$

  where α is some scaling constant.
- In practice, there are likely to be many possible distributions that satisfy the equation above within experimental certainty. We need an imaging system that, at the very least, can narrow the list of possibilities to a subset that has the same site occupancy in the site of interest

Comment: The reason this method might be worth the effort is that we are able to gain some information about site occupancy outside the site region of interest - this could help reduce occupancy uncertainty.

Conclusions (so far...)

Visibility

If we are confident that we will be able to image only a single plane of atoms, than we can see a strong improvement in visibility as we increase NA from 0.5 to 0.6. But, It doesn't do us much good to kill ourselves trying to improve our imaging beyond an NA=0.6.

If we think that we might have to image atoms with adjacent planes present, then imaging between NA=0.5 to NA=0.65 offers no improvement (not taking into account what the visibility variance is doing in this region). However, increasing the NA from 0.65 to 0.7 offers a two-fold improvement.

A question: what visibility is good enough? 10 percent? if so, a NA=0.45 objective is good enough for single-plane imaging, while a NA=0.6 objective is probably required for multi-plane imaging.

The multi-plane visibility variance remains to be calculated (it takes a while on Mathematica...a few days at least)

Threshold Identification

For single-plane identification, we should be fine with the NA=0.5 objective we already have. This gives us a greater than 99 percent calculated fidelity when collecting 100 photons per atom.

For multi-plane identification, we need to capture 1000 photons to be able to identify atoms with an NA=0.5 objective with greater than 99 percent calculated fidelity.

Matching and Minimizing

I'm not sure this is worth thinking about any further - at least not until we have atoms in a lattice and simpler methods have failed.

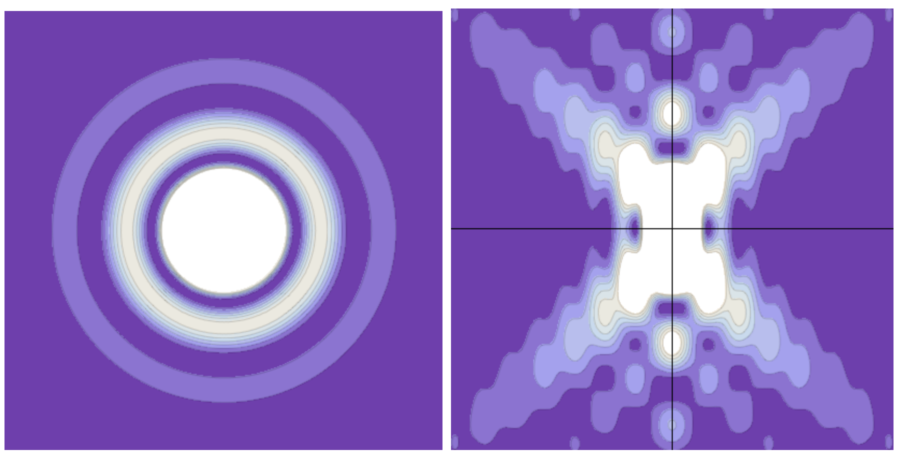


Figure 1: The image of a single point source as seen through a NA=0.6 lens (*left*) and the same image seen from the side (*right*).

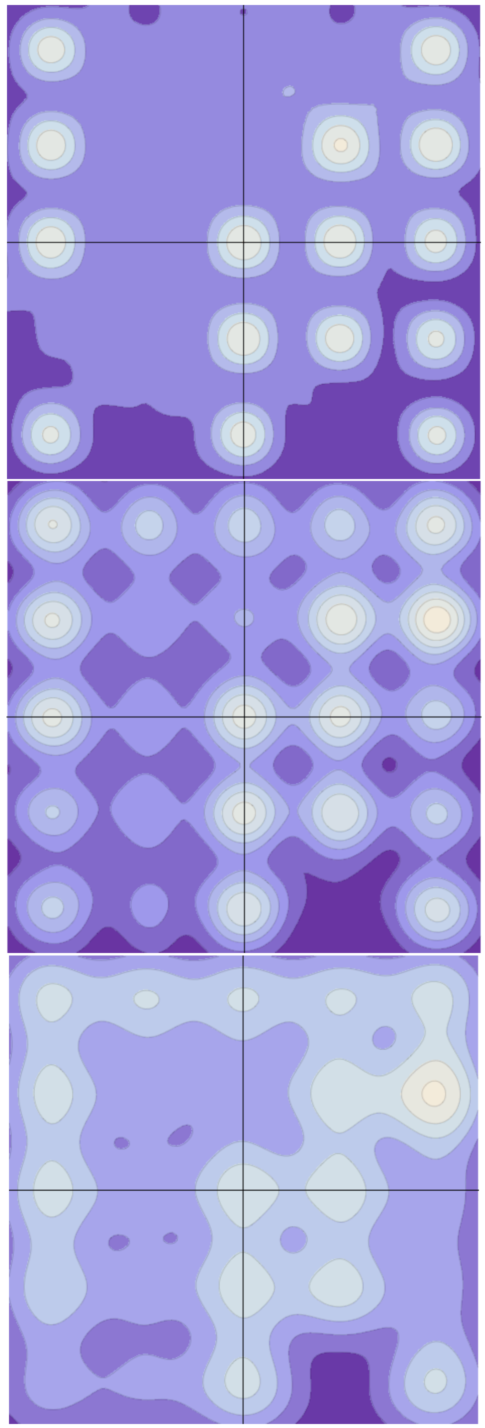


Figure 2: Expected images (including a top and bottom adjacent plane of atoms) as seen through (*from top to bottom*) an NA=0.6, NA=0.5, and NA=0.4 objective

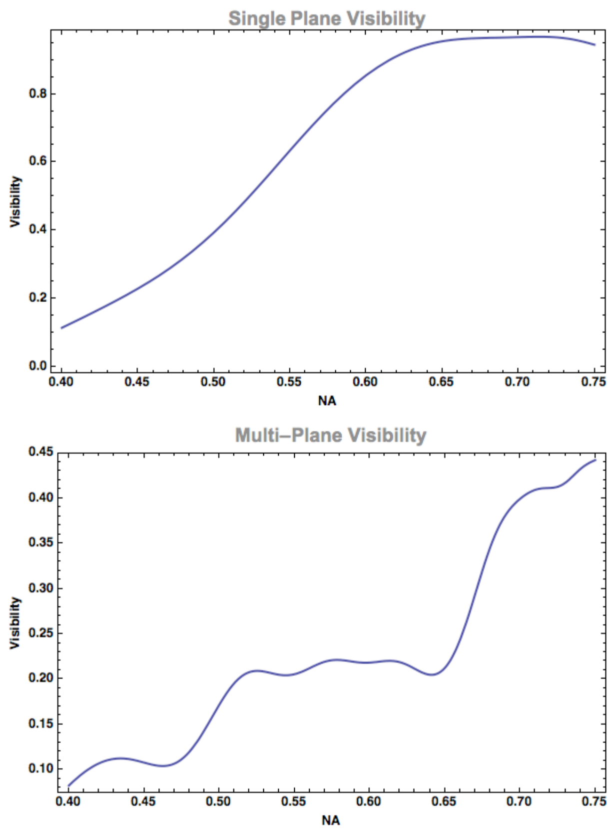


Figure 3: The visibility of an atom in a filled single lattice plane (*top*). The visibility of an atom in a filled multi-plane lattice (*bottom*).

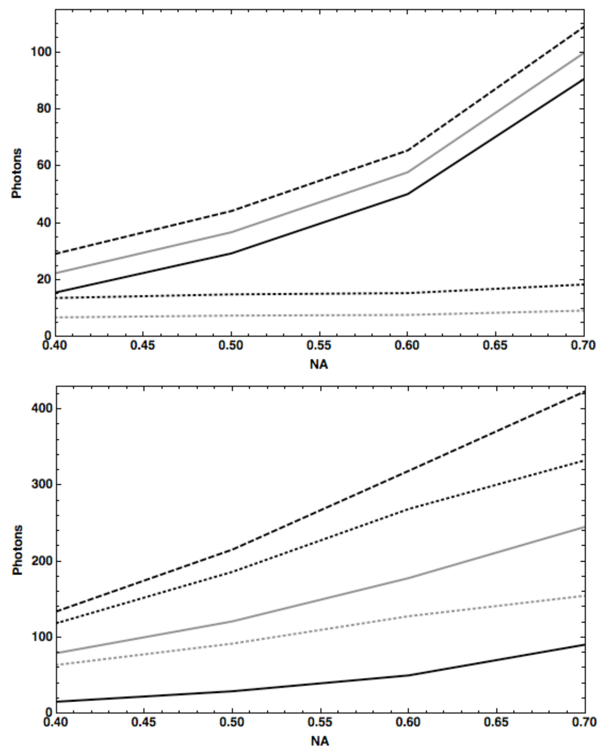


Figure 4: The Identification thresholds for atoms in a single plane (*top*) and in a multi-plane lattice (*bottom*): P_0 (dotted), P_{1a} (solid), and P_{1b} (dashed). The black lines represent a worst case occupation scenario and the gray lines represent an average occupation scenario.

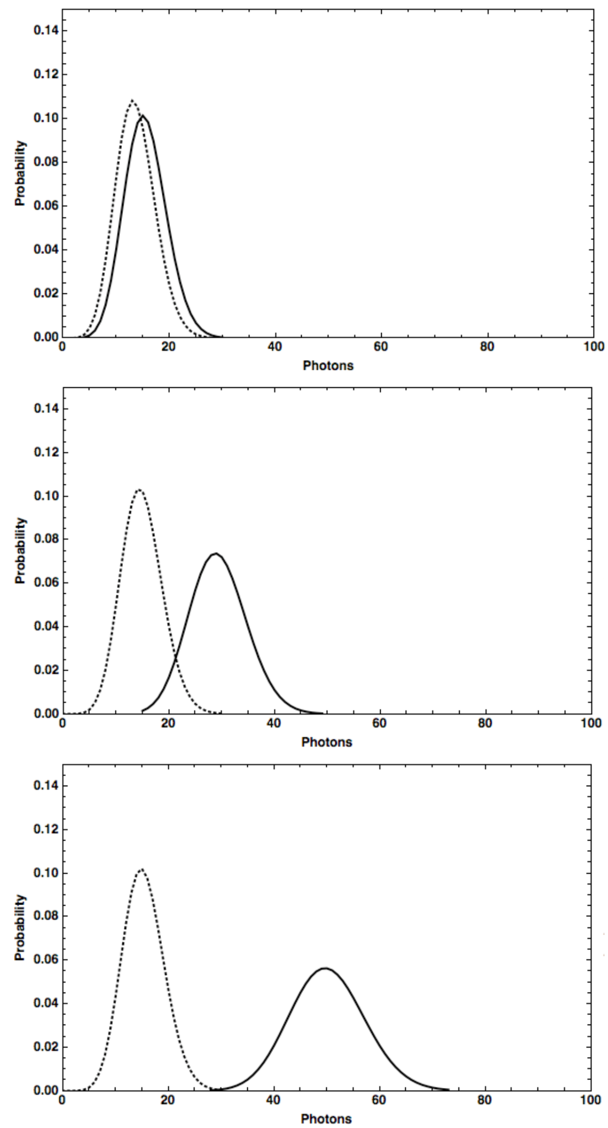


Figure 5: Histograms of worst case occupation scenarios for a single lattice plane as imaged with an NA=0.4 (*top*), NA=0.5 (*middle*), and NA=0.6 (*bottom*). We have only plotted P_0 (dotted) and P_{1a} (solid).

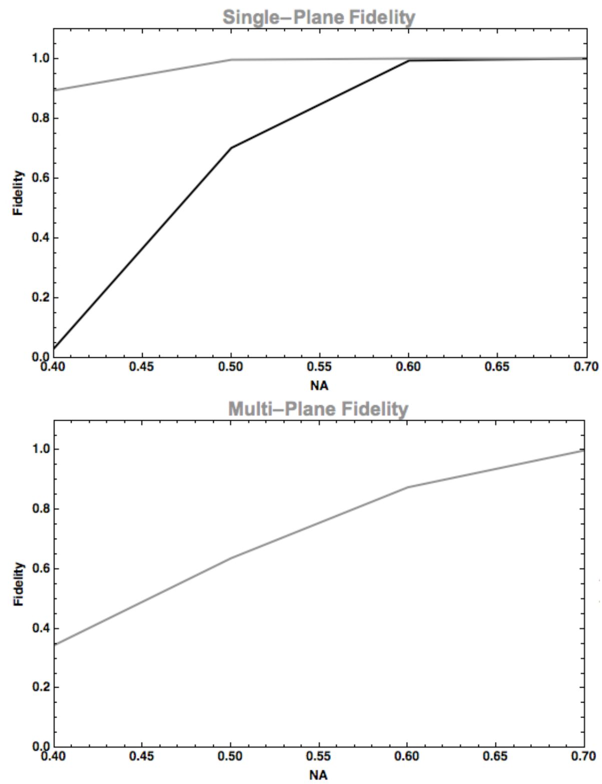


Figure 6: The Fidelity of determining the presence, or not, of an atom that has scattered 100 photons in a lattice site as a function of the imaging NA in a single occupied lattice plane (*top*) and through multiply occupied planes (*bottom*). The black lines represent a worst case occupation scenario and the gray lines represent an average occupation scenario. Note that the Muli-Plane Fidelity has no worst case scenario plotted - in fact the fidelity for this case is zero for all feasible NA when scattering 100 photons.

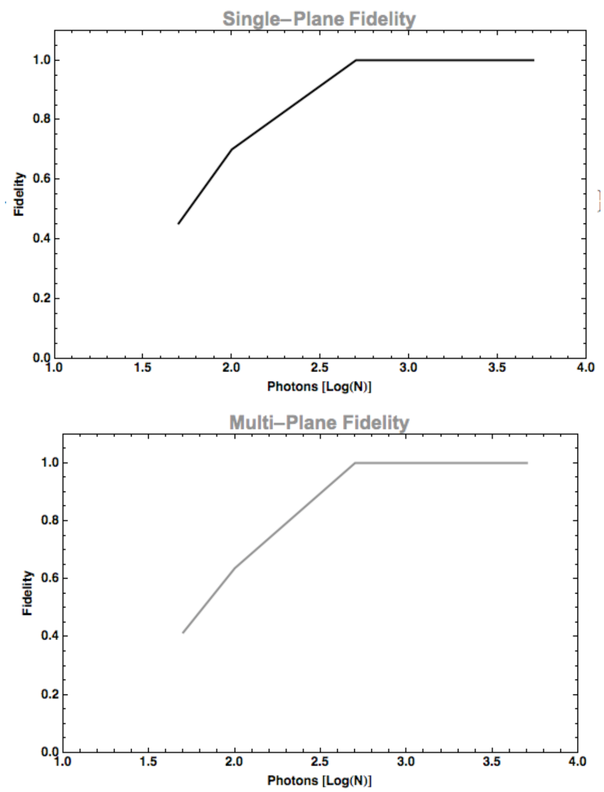


Figure 7: The Fidelity of determining the presence, or not, of an atom on a lattice site with a NA=0.5 objective as a function of the number of photons captured per atom in a single occupied lattice plane (*top*) and through multiply occupied planes (*bottom*). The black lines represent a worst case occupation scenario and the gray lines represent an average occupation scenario.